NTU JOURNAL OF PURE SCIENCES RESEARCH ARTICLE
EISSN: 2789-1097 Open Access

Lindely's method to estimate the parameters of the univariate truncated t Regression Model using informative prior information

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Received: 04-10-2021, Accepted: 04-11-2012, Published online: 05-12-2021

Abstract. In this research the parameters of the truncated t-regression model were estimated.in which the response variable follows a two-sided truncated t-distribution. Model's parameters were estimated by an approximate Bayesian technique according to Lindley's method. with unknown σ^2 and β. Informative prior information are used to estimate parameters base on the different ttruncating points and the degree of freedom which are known.

On the application side, experimental samples were generated by using the inverse function method. The application was carried out by relying on the Matlab program (14b) and according to different sample sizes, which are 10, 20 and 30 with degrees of freedom 3 and 6. Truncated points were selected from two sides , as well as for one term only, once from the left side and the other from the right side.The risk function was relied on in measuring the preference in relation to the sample size or the degree of freedom.

It was also concluded that the space of the truncated area leads to decrease the value of the risk function.

Key word : Lindely, t-truncated distribution , prior informative .regression ,loss fuction.

• Introduction

Many of the economic, medical, and agricultural studies, etc., are analyzed by regression models in which the error term follows a normal distribution. However, there are many cases where the error term belongs to the family of probability distributions with heavy tails such as the t-distribution and the generalized modified Bessel distribution [1]. These distributions are useful in reducing the impact of anomalous observations.

Another type of linear regression model was presented, which is the truncated normal regression model.In which either the response variable is a normal variable truncated from one or two sides, or the error term is a truncated variable. Many economic and medical phenomena that are defined on a part of the space of the normal variable have been analyzed by truncated normal regression models more efficiently than by analysis by untruncated normal regression models.

As it is known that the normal distribution is a special case of the t-distribution, and as long as the truncated normal distribution is included in the regression to cover the behavior of truncated normal phenomena. At the same time, the phenomena that have been studied as behaving in the behavior of a t-distribution such as prices and demand, and in the medical field such as myocardial infarction [2]. All these phenomena cannot be negative or very large in the positive direction, so it is necessary to analyze such phenomena using truncated t-regression models.

• Research objective:

The research aims are :

1)To estimate the parameters of some two-sided truncated t-regression models. The response variable was truncated in an approximate Bayes style by providing informative prior information using squared loss function.

2) To know the effect of the sample size on the truncation areas by knowing the path of the risk function, increase or decrease.

• Uses of the truncated t-distribution:

In practical applications [4] mentioned that it is possible to use truncated thick-tailed distributions in financial applications, physics,etc. [6] mentioned the multivariate t distribution is used in projection pursuit. [9] said that scientists began using distribution in the education environment.

• Truncated t distribution

It is known that the two-sided truncated probability distribution is within the period (a, b) , so that a < b takes the following form [8]:

$$f_T(\mathrm{x}) = \begin{cases} \dfrac{f(\mathrm{x})}{F(b)-F(a)} & a < x < b \\ 0 & other\ wise \end{cases} \quad (1)$$

whereas :

f (x) : represents the probability density function of the distribution.

F(b) – F(a): represents the cumulative function of the distribution.

Therefore, the probability density function of the truncated t-distribution of the random variable x within the period (a,b) is as follows:

$$f_T(\mathrm{x}) = \frac{f(\mathrm{x}|\mu,\sigma^2)}{\int_a^b f_v\,(\mathrm{x}|\mu,\sigma^2)dx} = \frac{f(\mathrm{x}|\mu,\sigma^2)}{F(b)-F(a)} \quad (2)$$

f (x) : represents the probability density function of t distribution with the degree of freedom v and the parameters of location and scale μ, σ^2, respectively, which takes the following form:

$$f(\mathrm{x}) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\ \Gamma\left(\frac{v}{2}\right)\sqrt{(\sigma^2)^{-1}}}\left(1 + \frac{(x-\mu)^2}{v\sigma^2}\right)^{-\frac{v+1}{2}} \quad -\infty < \mathrm{x}, \mu < \infty\ , \sigma^2 > 0 \quad (3)$$

And that Γ in equation (3) represents the complete gamma function.

F(b)-F(a): the cumulative function of t-distribution within the period (a, b) .

• Lindley's approximation

In many cases, it is difficult to find the posterior probability density function. The difficulty lies in finding the normalization constant or finding the posterior marginal distributions. On this basis, a Lindley approximation was presented to solve this difficulty. It is also called a bayes approximation estimator and is symbolized by the symbol u(θ), and the general form is [7]:

$$\hat{u}_B = u(\theta)|_{\theta=\hat{\theta}_{mle}} + \frac{1}{2}\sum_{i}^{m}\sum_{j}^{m}(u_{ij} + 2u_i\rho_j)\, s_{ij}|_{\theta=\hat{\theta}_{mle}} + \frac{1}{2}\sum_{i}^{m}\sum_{j}^{m}\sum_{k}^{m}\sum_{r}^{m} L_{ijk}s_{ij}s_{kl}u_l|_{\theta=\hat{\theta}_{mle}} + term\ of\ order\ n^{-2}\ or\ smaller \quad (4)$$

whereas :

m: depends on the number of parameters in the function. If θ is a value containing one parameter, m = 1 and when the vector contains m parameters, all upper limits of the sums are equal to m.

$\hat{\theta}$: represents a pre-estimator for the parameter vector of θ, some used the maximum likelihood estimator [5] .

ρ: the normal logarithm of the common prior distribution of the parameters of any model used, ρ_j: the derivative of the normal logarithm of p(θ) with respect to the parameter θ_j

$u(\underline{\theta})$: the parameter function θ, u_i: the first derivative of $u(\underline{\theta})$with respect to θ, u_{ij}: the second derivative $u(\theta)$with respect to θ_j , θ_i.

$L_{ijk} = \frac{\partial^3 lnL}{\partial\theta_i\partial\theta_j\partial\theta_k}$: It is the third derivative of the normal logarithm of the maximum likelihood function relative to the parameters $\theta_i, \theta_j, \theta_k$.

$s_{ij} = [-L_{ij}]^{-1}$: The element (i,j) of the negative inverse of the second derivative matrix of the likelihood function.

Now, the parameters will be estimated when σ^2 and $\underline{\beta}$ are unknown by using Lindley's approximation. It will be necessary to calculate several derivatives, as the index of the derivative 1 represents the derivative with respect to β_0, the index of the derivative 2 represents the derivative with respect to $\beta_1$and 3 represents the index of the derivative with respect to σ^2 .

The second and third partial derivatives are placed in two matrices, $[L^{(3)}], [L^{(2)}]$.

- Estimation by using informative prior information:

A simple linear truncated t regression mode is :

$$y_i = \beta_0 + \beta_1 x_i + u_i \quad \forall \ i = 1,2, \dots n \qquad (5)$$

By using mixed distribution, t-distribution is expressed as a mixture of the normal distribution and the inverse gamma distribution [10],then we will depend this formula to estimation .

The informative prior distributions will be used when the parameters $\beta_0, \beta_1, \sigma^2$ are unknown as follows :

Using Bayes' theorem, the joint prior distribution of $\underline{\beta}$and σ^2 conditioned by τ is:

$$P\left(\underline{\beta}, \sigma^2|\tau\right) \sim P\left(\underline{\beta}|\sigma^2, \tau\right) * P(\sigma^2|\tau) \qquad (6)$$

By substituting for the prior distribution of $\underline{\beta}$ and σ^2, and since the prior distribution with informative information for the parameter vector $\underline{\beta} = (\beta_0, \beta_1)$ conditioned by σ^2 and τ is as follows:

$$\left(\underline{\beta}|\sigma^2, \tau\right) \sim N_2\left(\underline{\beta_0}, \sigma^2 V_0\right)$$

whereas :

V_0 : a matrix with space (2*2), positive and symmetrical.

$\underline{\beta_0}$: vector with space (2 * 1)

$\underline{\beta}_0 = (\beta_{00} \quad \beta_{01})$,And the vector $\underline{\beta_0}$ and the matrix V_0 are imposed in the application by the researcher, or they are estimated in advance from previous data similar to the data of the study. And the prior probability density function for $\underline{\beta}|\sigma^2, \tau$ is as follows:

$$P\left(\underline{\beta}\,|\sigma^2, \tau\right) = \frac{1}{2\pi\tau\sigma^2|V_0|^{1/2}}\, e^{-\frac{(\underline{\beta}-\underline{\beta_0})' V_0^{-1}(\underline{\beta}-\underline{\beta_0})}{2\tau\sigma^2}} \qquad (7)$$

and the prior distribution of σ^2conditioned by τ is an gamma inverse distribution .It is described as follows:

$$(\sigma^2|\tau) \sim IG\left(\frac{a_0}{2}, \frac{b_0}{2}\right)$$

The prior probability density function for σ^2 is as follows

$$P(\sigma^2|\tau) = \frac{\left(\frac{b_0}{2}\right)^{\frac{a_0}{2}}}{\Gamma\left(\frac{a_0}{2}\right)} (\sigma^2)^{-\left(\frac{a_0}{2}+1\right)} e^{-\frac{b_0}{2\sigma^2}} \qquad (8)$$

Substituting for the prior distribution of $(\underline{\beta})$ (σ^2) and according to equations (7), (8), we get:

$P\left(\underline{\beta},\sigma^2|\tau\right)$

$$= \frac{1}{2\pi\tau\sigma^2|V_0|^{1/2}}\, e^{-\frac{\left(\underline{\beta}-\beta_0\right)' V_0^{-1}\left(\underline{\beta}-\beta_0\right)}{2\tau\sigma^2}}\left(\tau\sigma^2 V_0\right)^{-\left(\frac{a_0}{2}+1\right)} e^{-\frac{b_0}{2\sigma^2 V_0\tau}}$$

$$* \frac{\left(\frac{b_0}{2}\right)^{\frac{a_0}{2}}}{\Gamma\left(\frac{a_0}{2}\right)} \quad (9)$$

Whereas, equation (9) represents the joint probability distribution of the two parameters β,σ^2.

To estimate the parameters (β_0,β_1,σ^2), the terms of the sum will be in a Lindley approximation up to m=3 . After simplifying equation (4), we get the following:

By taking the first derivative of equation (11) for the parameters $(\beta_0,\beta_1,\sigma^2)$, we get the following

$$\rho = lnP\left(\underline{\beta}\,|\sigma^2,\tau\right) = -ln(2\pi\tau\sigma^2) - \frac{1}{2}ln|V_0| - \frac{\omega}{2\tau\sigma^2} - \left(\frac{a_0}{2}+1\right)ln(\sigma^2) - \frac{b_0}{2\sigma^2} + \frac{a_0}{2}\,ln\left(\frac{b_0}{2}\right) - ln\left(\Gamma\left(\frac{a_0}{2}\right)\right) \quad (11)$$

To facilitate the derivation process, we assumed that:

$$\omega = \left(\underline{\beta}-\beta_0\right)' v_0^{-1}\left(\underline{\beta}-\beta_0\right) \quad (12)$$

$$= \left[\begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix} - \begin{bmatrix}\beta_{00}\\ \beta_{10}\end{bmatrix}\right]' \begin{bmatrix}v_{00} & v_{01}\\ v_{10} & v_{11}\end{bmatrix}^{-1}\left[\begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix} - \begin{bmatrix}\beta_{00}\\ \beta_{10}\end{bmatrix}\right] \quad (13)$$

$$= \left[(\beta_0-\beta_{00}) \quad (\beta_1-\beta_{10})\right]\begin{bmatrix}v_{00} & v_{01}\\ v_{10} & v_{11}\end{bmatrix}^{-1}\begin{bmatrix}\beta_0-\beta_{00}\\ \beta_1-\beta_{10}\end{bmatrix}$$
$$= (\beta_0-\beta_{00})' v_{00}^*(\beta_0-\beta_{00}) + (\beta_1-\beta_{10})' v_{10}^*(\beta_0-\beta_{00}) + (\beta_0-\beta_{00})' v_{01}^*(\beta_1-\beta_{10}) + (\beta_1-\beta_{10})' v_{11}^*(\beta_1-\beta_{10}) \quad (14)$$

$$\hat{u}_\beta = u = (\hat{\theta})|_{\theta=\hat{\theta}_{mle}} + \frac{1}{2}\{(u_{11}s_{11} + u_{21}s_{21} + u_{31}s_{31} + u_{12}s_{12} + u_{22}s_{22} + u_{32}s_{32} + u_{31}s_{13} + u_{23}s_{23} + u_{33}s_{33}) + 2\,(u_1\rho_1 s_{11} + u_2\rho_1 s_{21} + u_3\rho_1 s_{31} + u_1\rho_2 s_{12} + u_2\rho_2 s_{22} + u_3\rho_2 s_{32} + u_1\rho_3 s_{13} + u_2\rho_3 s_{23} + u_3\rho_3 s_{33}) + \frac{1}{2}(L_{111}s_{11}s_{11}u_1 + L_{212}s_{21}s_{21}u_1 + L_{313}s_{31}s_{31}u_1 + L_{121}s_{12}s_{12}u_2 + L_{222}s_{22}s_{22}u_2 + L_{323}s_{32}s_{12}u_2 + L_{131}s_{13}s_{13}u_2 + L_{232}s_{23}s_{23}u_3 + L_{333}s_{23}s_{23}u_3)\Big\} \quad (10)$$

Since:

$v_0^{-1} = \left[v_{ij}^*\right] \quad i,j = 1,2 \quad (15)$

So:

$$\rho_1 = \frac{\partial ln\rho}{\partial \beta_0} = -\frac{\partial}{\partial \beta_0}\left(\frac{\omega}{2\tau\sigma^2}\right)$$
$$= \left(\frac{(\beta_0 - \beta_{00})v_{00}^* + v_{10}^*(\beta_1 - \beta_{10})}{\tau\sigma^2}\right) \; (16)$$

As well :

$$\rho_2 = \frac{\partial ln\rho}{\partial \beta_1} = -\frac{\partial}{\partial \beta_1}\left(\frac{\omega}{2\tau\sigma^2}\right) = \left(\frac{(\beta_1 - \beta_{10})v_{01}^* + v_{10}^*(\beta_0 - \beta_{00})}{\tau\sigma^2}\right) \quad (17)$$

$$\rho_3 = \frac{\partial ln\rho}{\partial \sigma^2} = \left(\frac{(3+a_0+b_0)\sigma^2}{2(\sigma^2)^2} + \frac{\omega}{2\tau(\sigma^2)^2}\right) \; (18)$$

By applying Lindley's equation to get the estimates for $\hat{\sigma}^2, \hat{\beta}_1, \hat{\beta}_0$ that were previously defined in equation (4), we get the following:

$$E\left(\hat{\beta}_{0B} \middle| data, \tau\right) = \hat{u}_{B|\tau} = \left[\beta_0 - \frac{n+1}{2\sigma^2}s_{13} + \frac{1}{2}\left[As_{11} + Bs_{21} + Cs_{31}\right]\right] \Big|_{\beta_0 = \hat{\beta}_0}$$

The unconditional bayesian estimator for β_0 is :

$$\hat{\beta}_{0B} = EE\left(\hat{\beta}_{0\beta} \middle| data, \tau\right)$$
$$= E\int_0^\infty (\hat{\beta}_{0B}|\tau) f(\tau) d\tau \qquad (19)$$

We integrate equation (19) with respect to τ, we get the unconditionally estimate $\hat{\beta}_{0B}$ we get the following result:

$$\hat{\beta}_{0_{B|\tau}} = \beta_0 + \frac{1}{2}\Bigg\{\sum_{i=1}^n \Bigg(2\,\sigma^2 \left(\frac{\left(\frac{v}{2}\right)^{v/2}\Gamma\left(\frac{v-2}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\left(\frac{v}{2}\right)^{\left(\frac{v-2}{2}\right)}}\right) E(y_i) + \left(x_i\underline{\beta}\right)E\left(y_i^2\right) - \sigma\left(\frac{\left(\frac{v}{2}\right)^{v/2}\Gamma\left(\frac{v-1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\left(\frac{v}{2}\right)^{\left(\frac{v-1}{2}\right)}}\right)\frac{b^2 f(\underline{b}) - a^2 f(\underline{a})}{F(\underline{b}) - F(\underline{a})}\Bigg)$$
$$+\left(\left(x_i\underline{\beta}\right)^2 + 2\sigma^2\left(\frac{\left(\frac{v}{2}\right)^{v/2}\Gamma\left(\frac{v-2}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\left(\frac{v}{2}\right)^{\left(\frac{v-2}{2}\right)}}\right) - 2\left(x_i\underline{\beta}\right)\sigma\left(\frac{\left(\frac{v}{2}\right)^{v/2}\Gamma\left(\frac{v-1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\left(\frac{v}{2}\right)^{\left(\frac{v-1}{2}\right)}}\right)\frac{f(\underline{b}) - f(\underline{a})}{F(\underline{b}) - F(\underline{a})}\right)^2$$
$$-2\left(\left(x_i\underline{\beta}\right) + \sigma\left(\frac{\left(\frac{v}{2}\right)^{v/2}\Gamma\left(\frac{v-2}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\left(\frac{v}{2}\right)^{\left(\frac{v-2}{2}\right)}}\right)\frac{f(\underline{b}) - f(\underline{a})}{F(\underline{b}) - F(\underline{a})}\right)\Bigg(\left(x_i\underline{\beta}\right)^2 + 2\sigma^2\left(\frac{\left(\frac{v}{2}\right)^{v/2}\Gamma\left(\frac{v-2}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\left(\frac{v}{2}\right)^{\left(\frac{v-2}{2}\right)}}\right) -$$
$$2\left(x_i\underline{\beta}\right)\sigma\left(\frac{\left(\frac{v}{2}\right)^{v/2}\Gamma\left(\frac{v-1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\left(\frac{v}{2}\right)^{\left(\frac{v-1}{2}\right)}}\right)\frac{f(\underline{b}) - f(\underline{a})}{F(\underline{b}) - F(\underline{a})}\Bigg) + 2\left(\left(x_i\underline{\beta}\right) + \sigma\left(\frac{\left(\frac{v}{2}\right)^{v/2}\Gamma\left(\frac{v-2}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\left(\frac{v}{2}\right)^{\left(\frac{v-2}{2}\right)}}\right)\frac{f(\underline{b}) - f(\underline{a})}{F(\underline{b}) - F(\underline{a})}\right)^3\Bigg\}$$
$$\Bigg\{2n - \sum_{i=1}^n\Bigg[\Bigg(\left(x_i\underline{\beta}\right)^2 + 2\sigma^2\left(\frac{\left(\frac{v}{2}\right)^{v/2}\Gamma\left(\frac{v-2}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\left(\frac{v}{2}\right)^{\left(\frac{v-2}{2}\right)}}\right) - 2\left(x_i\underline{\beta}\right)\sigma\left(\frac{\left(\frac{v}{2}\right)^{v/2}\Gamma\left(\frac{v-1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\left(\frac{v}{2}\right)^{\left(\frac{v-1}{2}\right)}}\right)\frac{f(\underline{b}) - f(\underline{a})}{F(\underline{b}) - F(\underline{a})}\Bigg)$$
$$-\left(\left(x_i\underline{\beta}\right) + \sigma\left(\frac{\left(\frac{v}{2}\right)^{v/2}\Gamma\left(\frac{v-2}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\left(\frac{v}{2}\right)^{\left(\frac{v-2}{2}\right)}}\right)\frac{f(\underline{b}) - f(\underline{a})}{F(\underline{b}) - F(\underline{a})}\right)^2\Bigg\}^{-2} \qquad (20)$$

To obtain the estimate B_1, we follow these steps:

$$E\left(\hat{\beta}_{1B} \middle| data, \tau\right) = \hat{u}_{B|\tau}$$
$$= \beta_1 - \frac{n+1}{2\sigma^2}s_{23}$$
$$+ \frac{1}{2}\left[As_{12} + Bs_{22} + Cs_{32}\right]$$

In order to get the unconditional estimator of the parameter β_{1B} , the above equation is integrated for the variable τ as follows:

$$\hat{\beta}_{1B} \simeq EE\left(\hat{\beta}_{1B} \middle| data, \tau\right) \simeq E\int_0^\infty (\hat{\beta}_{1B}|\tau) f(\tau) d\tau$$

$$\hat{\beta}_{1B} = \beta_1 + \frac{1}{2}\Bigg\{-\sum_{i=1}^n x_i^{\,2}\left[E\left(y_i^3\right) - \left(E\left(y_i^2\right)\right)^2 - 2E(y_i)E\left(y_i^2\right) + \right.$$

$$2\left(E(y_i)\right)^3 \Bigg]\left(-\left(\frac{\left(\frac{v}{2}\right)^{v/2}\Gamma\left(\frac{v-4}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\left(\frac{v}{2}\right)^{\left(\frac{v-4}{2}\right)}}\right)\frac{(\sigma^2)}{\sum_{i=1}^{n} x_i}\right)^2 \left(\left[E(y_i^2) - (Ey_i)^2\right]^{-1}\right)^2\Bigg|_{\beta_1=\hat{\beta}_1} \quad (21)$$

In the same way, we get the estimate σ_B^2 as follows:

$$E\ (\ \sigma_B^2\ |data, \tau\ ) =\ \hat{u}_{B|\tau}$$

$$=\ \sigma^2 - \frac{n+1}{2\sigma^2}s_{33} + \frac{1}{2}\left[As_{13} + Bs_{23} + Cs_{33}\right] \quad (22)$$

Where A,B,C as follows

$$A =\ s_{11}L_{111} + 2s_{12}L_{121} + 2s_{13}L_{131} + 2s_{23}L_{231} + s_{22}L_{221} + s_{33}L_{331} \quad (23)$$

$$B =\ s_{11}L_{112} + 2s_{12}L_{122} + 2s_{13}L_{132} + 2s_{23}L_{232} + s_{22}L_{222} + s_{33}L_{332} \quad (24)$$

$$C =\ s_{11}L_{113} + 2s_{12}L_{123} + 2s_{13}L_{133} + 2s_{23}L_{233} + s_{22}L_{223} + s_{33}L_{333} \quad (25)$$

By integrating on equation (22) with respect the variable τ, we get the unconditional Bayesian approximation for the parameter σ^2 as follows :

$$\sigma_B^2 \simeq EE\ (\ \sigma_B^2\ |data, \tau\ ) = E\ \int_0^\infty (\hat{\sigma}_B^2|\tau)f(\tau)d\tau$$

$$\hat{u}_B = u(\hat{\theta})|_{\theta=\hat{\theta}_{mle}} + \frac{1}{2}\ [(u_{11}s_{11} + u_{22}s_{22}) + (u_1\rho_1 s_{11} + u_2\rho_2 s_{22})] + + \frac{1}{2}\ [L_{111}s_{11}s_{11}u_1 + + L_{222}s_{22}s_{22}u_2] + term\ of\ order\ n^{-2}\ or\ smaller$$

$$\hat{u}_B = \hat{\sigma}^2 + \frac{1}{2}\left[\left(-2\left(\frac{n+1}{2\,\sigma^2}\right)\right)\left(\left(\frac{\left(\frac{v}{2}\right)^{v/2}\Gamma\left(\frac{v-2}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\left(\frac{v}{2}\right)^{\left(\frac{v-2}{2}\right)}}\right)(\sigma^2)^3 E\left(\sum_{i=1}^{n} y_i^2 + 2\,\beta_0\sum_{i=1}^{n} y_i + 2\beta_1\sum_{i=1}^{n} x_i y_i\right)^{-1} - \left(\frac{\left(\frac{v}{2}\right)^{v/2}\Gamma\left(\frac{v-2}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\left(\frac{v}{2}\right)^{\left(\frac{v-2}{2}\right)}}\right)(\sigma^2)^3\left(\sum_{i=1}^{n} E\left((y_i^2 - d_{1i}y_i)\right)\right)^{-1} + \left(2\left(\frac{\left(\frac{v}{2}\right)^{v/2}\Gamma\left(\frac{v-2}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\left(\frac{v}{2}\right)^{\left(\frac{v-2}{2}\right)}}\right)\right)^2(\sigma^2)^2\left(\sum_{i=1}^{n}\left[E\left((y_i^4 + 2d_{1i}y_i^3 + d_{3i}y_i^2 - d_{1i}d_{2i}y_i)\right) - E\left((y_i^2 - d_{1i}y_i)|\tau\right)E(((y_i^2 - d_{1i}y_i + d_{2i}))\right]\right)^{-1}\right)\right|_{\sigma^2=\hat{\sigma}^2} \quad (26)$$

- The squared risk function of the estimations of the two parameters of the truncated regression $\hat{\beta}_B$ and σ^2 :

The risk function for the parameter estimator $\hat{\beta}_B$ is as follows (Hormuz, 1999):

$$MSE(\theta) = E\left(\beta - \hat{\beta}_B\right)'\left(\beta - \hat{\beta}_B\right) \quad (27)$$

As for the risk function for the parameter estimator σ^2 as in the equation (28)

$$MSE(\sigma^2) = E(\ \sigma^2 - \ \hat{\sigma}_B^2)^2 \quad (28)$$

- Practical side
- Generate an empirical sample of the linear regression model:

The Monte Carlo simulation depends on random numbers, and a random number is a number whose probability of occurrence is equal to the probability of any other random number from a set of random numbers occurring period [0,1] .

The sample is generated in several ways, including:

1- Inverse function.

2- Convolution method.

It will be relied on to generate data:

1- Inverse Method

It is a method by which a random variable that follows a specific distribution is obtained to generate random numbers that follow that distribution, depending on the random numbers that follow the standard uniform probability distribution. As follows:

1- First finds the cumulative function.

2- Suppose the following cumulative function is :

$F(x) = P\{ y \leq x \}, 0 \leq F(x) \leq 1$

For all values of y defined within the space of the random variable

3- Then we do the following steps:

1- Generating random numbers R from a standard uniform distribution U(0,1).

2- Calculate or find the value of x from $x = F^{-1}(R)$

- Simulation sample experiment stage

The sample generation experiment included five stages:

The first stage :

In the first stage, It was assumed the following situation:

When $\sigma^2 = 0.5$ in and :

a) $\beta = [\beta_0 \quad \beta_1] = [1 \quad 4]$

b) $\beta = [\beta_0 \quad \beta_1] = [1 \ -4]$

With a choice of a set of different sample sizes, $n_1 = 10$, $n_2 = 20$, $n_3 = 30$, and 3 and 6 degrees of freedom .

The second stage :

This stage includes two steps:

The first step :

In this step, data was generated with a normal distribution with a mean equal to 4 and a variance equal to 0.5 to represent the values of the independent variable x, then these data were multiplied by default values of the two model parameters β_0 and β1 whose values are mentioned in the first stage.

The second step :

Generating random observations from u(0,1) with a size of 50 observations and then multiplying this data by the cumulative function of the truncated standard t distribution with the values (a - bx) and (b + bx), in order to obtain the error data that distributes t distribution for the period from (a – bx) to (b + bx), where the values of a and b represent different states: (0,3) , (0,2) , (0,1) , (-3,0) , (-2,0) , (-1,0) , (-3,3) , (-2,2), (-1,1)

The third step :

Observations of the response variable that follow truncated t-distribution with parameters (β_0, β_1) are obtained by summing the random error observations generated in the second step plus bx in the first step.

The third stage :

At this stage, the required sample size is deducted from the generated observations, and then we follow the following:

1- After generating the data, the heteroscedasticity of errors is tested according to Cold-Fields criterion.

2- The test for the lack of autocorrelation of the response variable, and the following figure shows that:

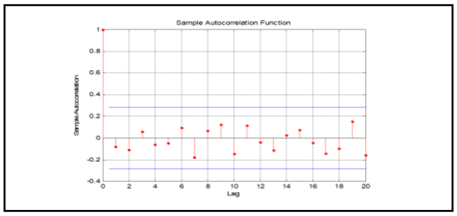

Figure (1): The autocorrelation of the response variable

Where it appears from Figure (1) that all data are located within the bounds of the two correlation lines, which indicates the randomness of the sample.

The following are the steps of applying the simulation stages to obtain the results according to the Lindley approximation:

Matlab (14b) was adopted to build the data generation program to obtain observations of the response and explanatory variables y and x, and equations (20), (21) and (26) were applied to find the estimated parameter values $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\sigma}^2$ According to the Lindley approximation, the equations have been applied as follows:

- Estimation using prior information when all parameters are unknown.

Table No. (1) in Appendix A shows the results obtained from applying equations (20), (21), (26), (27) and (28), as it appears from Table (1) the following:

When the sample size is n = 10 degrees of freedom v = 3, the path of the risk function fluctuates between descending and then rising when truncation from two sides or from the right side, while it decreases when truncation from the left side.

The fourth stage :

In this stage of the experiment, the parameters of the truncated regression model are estimated in a Bayesian style according to the equations previously mentioned and according to the sample sizes that were assumed in the first stage of the simulation experiment.

The fifth stage :

This stage included presenting the results to find the best estimate of the model parameters.

But when the degree of freedom is v = 6 and at the same size as the previous sample, the risk function increases at truncation from two sides and fluctuates when truncation from the right side, while it decreases when truncation from the left side.

When the sample size is n = 20, the risk function fluctuates when truncated from two sides and at degrees of freedom v = 3 and v = 6, while it decreases when truncated from one side and for both degrees of freedom 3 and 6.

But when the sample size is n = 30, the risk function decreases when truncated from two or one side and for degrees of freedom v = 3 and v = 6.

As for the risk function, in general, it decreases when the degrees of freedom increase, and because of the fluctuation in the value of the risk function at sample sizes 10 and 20, it is not clear that the risk function decreases when the sample size increases, but the effect of the sample size on the decreasing value of the risk function is more the greater the large sample size, as the risk function decreases when truncation from either side, in contrast to its value in other cases, as we mentioned above.

The risk function path can be illustrated in this case in the following table:

Table (1): The path of the risk function for the vector of regression parameters and σ^2,when the informative prior information is B=[1 -4] ,σ^2=0.5

sigma	beta	sample size	v = 3			v = 6		
			Truncated from two sides	Truncated from left	Truncated from right	Truncated from two sides	Truncated from left	Truncated from right
0.5	[1 4]	10	wiggle	decrease	wiggle	increase	decrease	wiggle
		20	wiggle	decrease	decrease	wiggle	decrease	decrease
		30	decrease	decrease	decrease	decrease	decrease	decrease

In the same way as before, we get the results when σ ^{2} = 0.5 , $\beta = [1 \ -4]$ which is shown in Appendix A (Table 2), which can be summarized in the following tables that show the path of the risk function for informative information, as follows:

Table (2): The risk function path for all parameter cases using informative information when σ^2 = 0.5 , $\beta = [1 \ -4]$

sigma	beta	the sample size	v = 3			v=6		
			Truncated from two sides	Truncated from left	Truncated from right	Truncated from two sides	Truncated from left	Truncated from right
0.5	[1 -4]	10	decrease	decrease	decrease	decrease	decrease	decrease
		20	decrease	decrease	decrease	decrease	decrease	decrease
		30	decrease	decrease	decrease	decrease	decrease	decrease

Table (2) shows that the risk function is always decreasing at sample sizes 10, 20 and 30 and degrees of freedom 3 and 6 using the informative informative.

- **Conclusions**:

1- When σ^2, β is unknown, it turns out that their estimations depend on the zero moments of the sixth order of the response variable.

2- On the application side, the risk function decreased with increasing degrees of freedom.

3- The risk function decreases as the sample size increases.

4- The amount of the risk function fluctuates up and then down or down and then up if the truncation is from two sides or from one side when the sample size is less than 30.

5-The risk function is always decreasing and there is no fluctuation if the truncation is from two sides or from the right or left side when the sample size is 30.

- **Recommendations**

Through the results obtained, you can check:

1-Using Lin's approximation to approximate the truncated probability density function , which is possible to obtain the posterior probability distributions of the parameters of the regression model and their estimators analytically.

2- Paying attention to Bayesian prediction in truncated normal and truncated t linear regression models because of its importance in decision-making.

- **References**

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Appendix A

Table No. (1): Bayesian estimation for the vector of regression parameters and σ^2 when the prior information is informative $[\sigma^2 = 0.5 , \beta = [\,1\ \ 4\,]]$

σ^2	β	SAMPLE SIZE	TRUNCATION POINTS (A,C)	V = 3				V = 6			
				$\hat{\beta}_{0B}$	$\hat{\beta}_{1B}$	$\hat{\sigma}^2_B$	Risk function	$\hat{\beta}_{0B}$	$\hat{\beta}_{1B}$	$\hat{\sigma}^2_B$	Risk function
0.5	[1 4]	n=10	(-1,1)	0.126	0.744	0.519	10.720	0.101	0.573	0.492	10.729
			(-2,2)	0.082	0.588	0.492	20.875	0.228	0.199	0.614	15.738
			(-3,3)	0.156	0.124	0.448	16.845	0.129	0.970	0.500	23.204
			(-1,0)	0.195	0.399	0.505	18.989	0.505	0.215	0.338	16.581
			(-2,0)	0.089	0.126	0.546	16.819	0.690	0.155	0.480	15.726
			(-3,0)	0.491	0.163	0.337	16.584	0.119	0.277	0.511	14.314
			(0,1)	0.287	0.161	0.433	18.112	0.312	0.236	0.518	17.514
			(0,2)	0.190	0.253	0.531	17.708	0.085	0.460	0.753	13.890
			(0,3)	0.168	0.362	0.567	18.709	0.601	0.149	0.590	15.702
0.5	[1 4]	n = 20	(-1,1)	0.196	0.176	0.518	17.199	0.188	0.258	0.492	17.144
			(-2,2)	0.025	1.390	0.626	15.805	0.110	0.945	0.624	11.872
			(-3,3)	0.307	0.425	0.490	18.536	0.194	0.199	0.545	17.322
			(-1,0)	0.374	0.317	0.157	17.105	0.330	0.153	0.246	15.661
			(-2,0)	0.190	0.166	0.501	15.502	0.583	0.390	0.861	14.713
			(-3,0)	0.517	0.222	0.191	14.350	0.015	0.925	0.611	10.133
			(0,1)	0.317	0.169	0.513	17.700	0.097	0.275	0.551	17.202
			(0,2)	0.199	0.351	0.261	17.423	0.231	0.239	0.445	16.781
			(0,3)	0.190	0.472	0.580	16.100	0.331	0.139	0.384	15.740
0.5	[1 4]	n=30	(-1,1)	0.114	0.681	0.590	20.767	0.130	0.551	0.574	19.250
			(-2,2)	0.160	0.391	0.567	18.198	0.255	0.110	0.441	15.557
			(-3,3)	0.119	0.190	0.552	16.241	0.178	0.090	0.600	15.270
			(-1,0)	0.867	0.198	0.356	16.315	0.167	0.181	0.444	16.217
			(-2,0)	0. 174	0.291	0.955	14.451	0.680	0.154	0.160	13.790
			(-3,0)	0.193	0.274	0.613	14.185	0.127	0.554	0.599	12.600

(0,1)	0.190	0.276	0.496	17.811	0.333	0.430	0.590	16.100
(0,2)	0.150	0.180	0.505	16.324	0.085	0.460	0.753	15.892
(0,3)	0.691	0.259	0.680	14.602	0.120	0.325	0.486	14.183

Table No. (2): Bayesian estimation for the vector of regression parameters and σ^2 when the prior information is informative $[\sigma^2 = 0.5\ , \beta = [\ 1\ \ -4\ ]]$

SIGMA	BETA	SAMPLE SIZE	TRUNCATION POINTS (A,C)	V = 3 $\hat{\beta}_{0B}$	V = 3 $\hat{\beta}_{1B}$	V = 3 $\hat{\sigma}^2_B$	V = 3 Risk function	V = 6 $\hat{\beta}_{0B}$	V = 6 $\hat{\beta}_{1B}$	V = 6 $\hat{\sigma}^2_B$	V = 6 Risk function
0.5	[1 -4]	n=10	(-1,1)	2.447	0.055	0.498	18.537	0.143	-0.003	0.518	16.711
			(-2,2)	0.244	-0.025	0.453	16.374	0.375	-0.144	0.519	15.260
			(-3,3)	0.043	-0.203	0.498	15.333	0.421	-0.175	0.500	14.966
			(-1,0)	0.138	-0.049	0.522	16.354	0.235	-0.034	0.478	16.315
			(-2,0)	0.038	-0.151	0.502	15.740	0.335	-0.166	0.498	15.142
			(-3,0)	1.224	-0.052	0.446	15.640	1.324	-0.148	0.466	14.944
			(0,1)	2.560	0.026	0.499	18.642	0.34	-0.112	0.524	15.553
			(0,2)	0.216	0.050	0.497	17.017	0.333	-0.25	0.517	14.508
			(0,3)	0.690	0.070	0.478	16.661	0.79	-0.27	0.498	13.957
0.5	[1 -4]	n = 20	(-1,1)	0.650	0.099	0.488	16.924	0.421	-0.025	0.52	16.136
			(-2,2)	0.265	-0.056	0.499	16.095	0.344	-0.175	0.473	15.062
			(-3,3)	1.249	-0.127	0.488	15.062	1.457	-0.255	0.518	14.234
			(-1,0)	0.361	-0.053	0.493	15.987	0.117	-0.111	0.500	15.904
			(-2,0)	0.938	-0.056	0.499	15.559	0.471	-0.147	0.513	15.126
			(-3,0)	0.999	-0.423	0,493	12.795	1.560	-0.623	0,493	11.718
			(0,1)	0.579	-0.023	0.444	15.997	1.700	-0.226	0.519	14.733
			(0,2)	0.679	-0.177	0.464	14.720	0.240	-0.312	0.504	14.179
			(0,3)	0.876	-0.421	0.432	12.829	0.991	-0.510	0.501	12.180
0.5	[1 -4]	n = 30	(-1,1)	1.022	-0.031	0,453	15.756	1.349	-0.073	0.508	15.543
			(-2,2)	1.136	-0.063	0.426	15.524	1.236	-0.137	0.446	14.981
			(-3,3)	1.122	-0.169	0,453	14.694	0.75	-0.299	0.508	13.760
			(-1,0)	0.217	-0.089	0.52	15.909	0.945	-0.102	0.518	15.198
			(-2,0)	0.855	-0.098	0.498	15.247	1.038	-0.144	0.519	14.871

(-3,0)	0.518	-0.647	0.521	11.475	0.418	-0.847	0.501	10.280
(0,1)	0.912	-0.123	0.499	15.039	0.431	-0.212	0.498	14.673
(0,2)	0.972	-0.301	0.981	13.915	0.765	-0.354	0.454	13.351
(0,3)	0.707	-0.968	0.500	9.279	0.807	-0.768	0.52	10.483