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Some Results on AWN-injective Rings

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ABSTRACT

A concept of AWN-injective ring is defined by [1], that is, for any $a \in N(D)$, there exists $n \ge 1$ and an Y-sub module X_a of μ (μ is aright D-module) such that $a^n \ne 0$ and $l_\mu r_R(a^n) = \mu a^n \oplus X_{a^n}$ as left S-module with S=End(μ). If D_D is AWN-injective module, then we call D aright AWN-injective ring. In this note also continue to study some extensions of AWN-injective rings. A mong others it is proved that, if D is aright AWN-injective ring, so is eDe for $e^2 = e \in R$ satisfying DeD = e. Also prove that D is reduced, and $GW\pi$ -reguler ring, if D is a weak symmetric and whose every simple right D-module is AWN-injective ring.



Introduction

Throughout this paper all rings are associative with unit, and D-module are unitial. For $a \in D$, r(a) and l(a) denote the right and left annihilator of a, respectively. We write Y(D), N(D), J(D) for the right singular ideal the set of nilpotent elements and Jacobson radical of D. Generalization of Ninjectivity (WN-injective) have been discussed in many papers [2,3,4]. Aright D-module μ is called N-injective ring (WN-injective), if $a \in N(D)$, $lr(a) = Da(lr(a^n) = Da^n)$. In [5] Zhao and DUintroduced an AN-injective module. Let μ be a right D-module with S=End (μ_D) . The module μ is called AN-injective, if for any $a \in N(D)$, there exists an Y-submodule X_a of μ such that $l_{\mu}r_D(a) =$ $\mu a \oplus X$ as left S-module. They also studied right AN-injective rings and gave some characterizations and properties which generalized results of [3]. The module μ called AGP-injective if, for any $a \in D$ there exists $n \ge 1$ and an Y-submodule X_a of μ such that $a^n \ne 0$ and $l_{\mu}r_D(a^n) =$ $\mu a^n \oplus X_a$ as left Y-module [6]. Right D-module μ is called JCPI [7,8] if, for any $a \notin Y(D)$ and any right D-homomorphism of aD into μ extends to one of D into μ . D is called right JCPI, if D is an JCPI module. A ring D is regular if for every $a \in D$ there is $b \in D$ so that a = aba [9]. In [10], the regular element in non-commutative rings have been discussed for example, $M_2(Z_2)$ and $M_2(Z_3)$. Also, [11] studied some properties of regular and simple generalized m-flat rings. D is N-regular if for all $a \in N(D)$ is regular [2].

2. Almost WN-injective rings

In [1], introduced the notion of almost WN-injective rings as a generalization of that of almost N-injective rings, namely a *module* μ is called almost WN-injective (AWNI) if for any $a \in N(D)$, there exists $n \ge 1$ and an Y-submodule X_a of μ such that $a^n \ne 0$ and $l_{\mu}r_D(a^n) = \mu a^n \oplus X_{a^n}$ as left S-module with S=End(μ). If D is a right AWNI, the we call D is a right AWNI-ring.

Many of the results on right AGP-injective rings were obtained for the class of AWNI- rings.

Example: Let D = Z is AWNI- ring but not AGP-injective.

Remark: Obviously, right WN-injective rings are right AWNI, but the convers is false in general. For example, let $D = \begin{bmatrix} 0 & A \\ 0 & A \end{bmatrix}$ with A is afield, $N(D) = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$. Let $0 \neq z \in A$. Then $lr \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & A \\ 0 & A \end{bmatrix}$ and $D \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$. Therefore $lr \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} \neq D \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}$, D is not right WN-injective. Note that $lr \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} = D \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & A \\ 0 & A \end{bmatrix}$. Therefore, R is right AWNI.

We now start with a lemma to be used extensively in the paper.

- **2.1 Lemma:**[3] Suppose that μ is a right D-module with S=End (μ_D) if $lr(a^n) = \mu a^n \oplus X_{a^n}$, where X_{a^n} is a left Y-submodule of μ . Set $f: a^n D \to \mu$ is a right D-homomorphism, then $f(a^n) = ma^n + x$, $m \in \mu, x \in X_a$.
- **2.2 Theorem:** If *D* is aright AWNI-ring, then eDe is AWNI-ring, $e \in Id(D)$ such that DeD = D.

Proof: Let S = eDe and $a \in N(S) = eNe$. Then $a = ae \in N(D)$, so there exists $n \ge 1$ and a left ideal X_{a^n} of D, such that $l_D r_D(a^n) = Da^n \oplus X_{a^n}$. Since $1 - e \in r(a^n)$ we see that t(1 - e) = 0 for any $t \in X_{a^n}$, which implies that $X_{a^n} = X_{a^n}e$. Thus $eDa^n e \cap eX_{a^n}e = 0$. Clearly, $eDa^n e \subseteq l_s r_s(a^n)$ and $eX_{a^n}e \subseteq l_s r_s(a^n)$. Since $Da^n e = Da^n$ and $X_{a^n} = X_{a^n}e$. Now, take $x \in l_s r_s(a^n)$ and write $1 = \sum_{i=1}^m a^n_i \ ebi$ for some $a_i, b_i \in D$ and $m \ge 1$. Then for any $y \in r(a^n)$, we get $a^n eya^n_i e = a^n ya^n_i e = 0$. This implies that $xeya^n_i e = 0$ for each I, which gives $xy = xey\sum_{i=1}^m a^n_i eb_i = 0$, since $x \in S$. So $x \in l_D r_D(a^n)$ and hence $l_s r_s(a^n) \subseteq l_D r_D(a^n)$. Take x = s + t, where $s \in Da^n$ and $t \in X_{a^n}$. Hence $x = exe = ese \ xete \in eRa^n e + eX_{a^n}e$ and $l_s r_s(a^n) = eDa^n e \oplus eX_{a^n}e = Sa^n \oplus eX_{a^n}$ where eX_{a^n} is aleft ideal of S. Therefore, S is right AWNI.

A ring D is called weak symmetric if abc being nilpotent implies that acb is nilpotent for all $a, b, c \in D$ [12].

- **2.3 Lemma:** [12] If D be a weak symmetric ring, then all nilpotent elements of D are in J(D).
- **2.4 Proposition:** Let D be weak symmetric and right AWNI- ring. Then $I(D) \subseteq Y(D)$.

Proof: Let $0 \neq \alpha \in N(D)$. If $\alpha \notin Y(D)$, then there exists anon zero ideal of D such that $r(\alpha) \cap K = 0$. Hence there exists $b \in K$ such that $\alpha b \neq 0$. Note that $\alpha \in J(D)(\text{Lem.2.3})$ and $\alpha b \in J(D)$. Since D is AWNI-ring, then there exists $0 \neq u \in \alpha bD$ and $n \geq 1$ such that $u^n \neq 0$ and $lr(u^n) = Du^n \oplus X$ for some a left ideal X_{u^n} of D. Write $u^n = \alpha bc$ for some $c \in D$. If $t \in r(\alpha bc)$, then $\alpha bct = 0$ and this implies that $ct \in r(\alpha b) = r(b)$ since $r(\alpha) \cap K = 0$. Hence (bc)t = b(ct) = 0 and so $t \in r(bc)$ so $r(bc) = r(\alpha bc)$, Note that $bc \in lr(bc) = lr(\alpha bc) = Du^n \oplus X_{u^n}$. Write $bc = d\alpha bc + x$, where $d \in D$. Hence $x = bc - d\alpha bc$, $x = (1 - d\alpha)bc$, $x(1 - d\alpha)^{-1} = bc$, and so $bc \in X_{u^n}$. Since $1 - d\alpha$ is invertible, contradicting with $d\alpha bc \in Du^n - X_{u^n}$. There fore $\alpha \in Y(D)$ and hence $J(D) \subseteq Y(D)$.

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2.5 Lemma:[2] Let *D* be a right JCPI. Then $Y(D) \subseteq J(D)$.

From Lem.(2.5) and prop.(2.4), we have the following corollary.

- **2.6 Corollary**: Let *D* be weak symmetric, right JCPI and right AWNI- ring, then I(D) = Y(D).
- **2.7 Theorem**: If D is a weak symmetric ring whose every simple right D-module is AWNI, then D is reduced.

Proof: Assume that $0 \neq a \in D$ with $a^2 = 0$. Thus $r(a) \subseteq L$, where L be amaximal right ideal of D. Since D/L is AWNI, then $l_{D/L}r(a^n) = D/L \oplus X_{a^n}$, $X_{a^n} \leq D/L$. Let $f: aD \to D/L$ be defined by $f(ay) = y + L, y \in D$. Note that f is well defined. So $1 + L = f(a) = ca + L, c \in D, x \in X_{a^n}$, by Lem.(2.1) $1 - ca + L = x \in D/L \cap X = 0$, so $1 - ca \in L$, since $ca \in N(D), 1 - ca$ is invertible, which is a contradiction. Therefore, D is a reduced ring.

According to [13], a ring D is called GW π - regular, if for all $a \in J(D)$ there exists $n \ge 1$ such that $a^n \in a^n D a^n D$.

2.8 Proposition: If D is a weak symmetric ring whose every simple right D-module is AWNI, then D is $GW\pi$ - regular ring.

Proof: Let $a \in N(D)$, then by Lem.(2.3), $a \in J(D)$. We will show that $Da^nD + r(a^n) = D$, $n \ge 1$. If $Da^nD + r(a^n) \ne D$, then there exists amaximal right ideal L of D containing $Da^nD + r(a^n)$. Then by a similar way as in the previous process, there exists $b \in D$ such that $1 - ba^n \in L$. Since $ba^n \in N(D)$, $1 - ba^n$ is invertible, which is a contradiction. There fore $Da^nD + r(a^n) = D$ for any $a \in J(D)$ and implies that D is $GW\pi$ - regular ring.

From Theorem (2.7) we get corollary

- **2.9 Corollary**: If D is a weak symmetric ring whose every simple right D -module is AWNI, then:
 - 1. $N(D) \cap J(D) = 0$
 - 2. If I(D) is nil, then I(D) = 0
- **2.10 Proposition:** Let D be a weak symmetric ring, Consequently, the subsequent conditions are similar. D is reduced.
 - 1. *D* is reduced.
 - 2. *D* is N-regular.
 - 3. *D* is a right N-injective.
 - 4. *D* is a right WN-injective.
 - 5. *D* is a right AWNI.
 - 6. Every simple right *D-module* is AWNI.

Proof: $1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 6$ is clear and by Theorem (2.7) $6 \rightarrow 1 \blacksquare$

2.11 Theorem: Let D be a weak symmetric and every simple singular right D-module is AWNI, then Y(D) = 0.

Proof: Suppose that $Y(D) \neq 0$, then Y(D) contains a non zero element z such that $z^2 = 0$. Therefore $r(z) \neq D$. Let L be a maximal right ideal of D containing r(z), then L is an essential right ideal of D which implies that $D \mid L$ is a right AWNI and $l_{D/L}r(z) = (D/L) \oplus X_z$, $X_z \leq D/L$. Let $f: zD \to D/L$ be defined by f(zy) = y + L, it is note that f is well defined. Hence, 1 + L = f(z) = cz + L + x, $c \in D$, $x \in X_z$, $1 - ca + L = xD/L \cap X = 0$. So $1 - cz \in L$. By Lem.(2.3), $N(D) \subseteq J(D)$, then $cz \in J(D) \subseteq L$. Hence $1 \in L$, contradicting that $L \neq D$. This proves that Y(D) = 0.

In [13], the trivial extension of any ring D, $D\alpha D = \{(d,c): d,c \in D\}$ with addition defined component vise and multiplication defined by (d,c)(b,y) = (db,dy+cb).

2.12 Proposition: Let D be a ring. If $D\alpha D$ is aright AWNI. Then D is right AGP-injective.

Proof: Let $W = D\alpha D$. For $d \in D$, $(0,d) \in N(W)$. So there exists a left ideal $X_{(0,d)}$ of W and $n \ge 1$ such that $(0,d)^n \ne 0$ and $l_W r_W(0,d)^n = W(0,d)^n \oplus X_{(0,d)^n}$. Since $(0,d)^2 = 0$. It must be that n=1. Hence $l_W r_W(0,d) = W(0,d) \oplus X_{(0,d)}$. By the proof of [14, Prop.(3.1)], if $(b,c) \in l_W r_W(0,d)$ and $(m,n) \in W(0,d)$ then b=0,m=0. So $X_{(0,d)}=0$ αX_d , where X_d is a left ideal of D. Again by [14, Prop.(3.1)] we have $l_D r_D(d) = Dd \oplus X_d$. Hence D is right AGP-injective.

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