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Some Results on AWN-injective Rings

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A B S T R A C T

A concept of AWN-injective ring is defined by [1], that is, for any $a \in N(D)$, there exists $n \geq 1$ and an Y-*sub module* X_a of μ (μ is aright *D-module*) such that $a^n \neq 0$ and $l_\mu r_R(a^n) = \mu a^n \oplus X_{a^n}$ as left S-*module* with S=End(μ). If D_D is AWN-injective *module*, then we call D aright AWN-injective ring. In this note also continue to study some extensions of AWN-injective rings. A mong others it is proved that, if D is aright AWN-injective ring, so is eDe for $e^2 = e \in R$ satisfying $DeD = e$. Also prove that D is reduced, and GWπ-reguler ring, if D is a weak symmetric and whose every simple right D *-module* is AWN-injective ring.

Introduction

Throughout this paper all rings are associative with unit, and D-*module* are unitial. For $a \in D$, $r(a)$ and $l(a)$ denote the right and left annihilator of a, respectively. We write $Y(D)$, $N(D), J(D)$ for the right singular ideal the set of nilpotent elements and Jacobson radical of D. Generalization of N-injectivity (WN-injective) have been discussed in many papers [2,3,4]. Aright D-*module* μ is called N-injective ring (WN-injective), if $a \in N(D)$, $lr(a) = Da(lr(a^n) = Da^n)$. In [5] Zhao and DU introduced an AN-injective *module*. Let μ be a right D-*module* with S=End (μ_D). The *module* μ is called AN-injective, if for any $a \in N(D)$, there exists an Y-*submodule* X_a of μ such that $l_\mu r_D(a) = \mu a \oplus X$ as left S-*module*. They also studied right AN-injective rings and gave some characterizations and properties which generalized results of [3]. The *module* μ called AGP-injective if, for any $a \in D$ there exists $n \geq 1$ and an Y-*submodule* X_a of μ such that $a^n \neq 0$ and $l_\mu r_D(a^n) = \mu a^n \oplus X_a$ as left Y-*module* [6]. Right D-*module* μ is called JCPI [7,8] if, for any $a \notin Y(D)$ and any right D-homomorphism of aD into μ extends to one of D into μ. D is called right JCPI, if D is an JCPI *module*. A ring D is regular if for every $a \in D$ there is $b \in D$ so that $a = aba$ [9]. In [10], the regular element in non-commutative rings have been discussed for example, $M_2(Z_2)$ and $M_2(Z_3)$. Also, [11] studied some properties of regular and simple generalized m-flat rings. D is N-regular if for all $a \in N(D)$ is regular [2].

2.Almost WN-injective rings

In [1], introduced the notion of almost WN-injective rings as a generalization of that of almost N-injective rings, namely a *module* μ is called almost WN-injective (AWNI) if for any $a \in N(D)$, there exists $n \geq 1$ and an Y-*submodule* X_a of μ such that $a^n \neq 0$ and $l_\mu r_D(a^n) = \mu a^n \oplus X_{a^n}$ as left S-*module* with S=End(μ). If D is a right AWNI, the we call D is a right AWNI-ring.
Many of the results on right AGP-injective rings were obtained for the class of AWNI- rings.

Example: Let $D = Z$ is AWNI- ring but not AGP-injective.

Remark: Obviously, right WN-injective rings are right AWNI, but the convers is false in general. For example, let $D = \begin{bmatrix} 0 & A \\ 0 & A \end{bmatrix}$ with A is afield, $N(D) = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$. Let $0 \neq z \in A$. Then $lr\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & A \\ 0 & A \end{bmatrix}$ and $D\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$. Therefore $lr\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} \neq D\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}$, D is not right WN-injective. Note that $lr\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} = D\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & A \\ 0 & A \end{bmatrix}$. Therefore, R is right AWNI.
We now start with a lemma to be used extensively in the paper.

Lemma (2.1):[3] Suppose that μ is a right D-*module* with S=End (μ_D) if $lr(a^n) = \mu a^n \oplus X_{a^n}$, where X_{a^n} is a left Y-*submodule* of μ. Set $f: a^n D \longrightarrow \mu$ is a right D-homomorphism, then $f(a^n) = ma^n + x, m \in \mu, x \in X_a$.

Theorem (2.2): If D is aright AWNI- ring, then eDe is AWNI-ring, $e \in Id(D)$ such that $DeD = D$.

Proof: Let $S = eDe$ and $a \in N(S) = eNe$. Then $a = ae \in N(D)$, so there exists $n \geq 1$ and a left ideal X_{a^n} of D, such that $l_D r_D(a^n) = Da^n \oplus X_{a^n}$ .Since $1 - e \in r(a^n)$ we see that $t(1-e) = 0$ for any $t \in X_{a^n}$, which imiplies that $X_{a^n} = X_{a^n}e$. Thus $eDa^n e \cap eX_{a^n}e = 0$. Clearly, $eDa^n e \subseteq l_s r_s r(a^n)$ and $eX_{a^n}e \subseteq l_s r_s(a^n)$. Since $Da^n e = Da^n$ and $X_{a^n} = X_{a^n}e$. Now, take $x \in l_s r_s(a^n)$ and write $1 = \sum_{i=1}^{m} a^n{}_i\ ebi$ for some $a_i, b_i \in D$ and $m \geq 1$. Then for any $y \in r(a^n)$, we get $a^n eya^n{}_i e = a^n ya^n{}_i e = 0$. This implies that $xeya^n{}_i e = 0$ for each I, which gives $xy = xey \sum_{i=1}^{m} a^n{}_i eb_i = 0$, since $x \in S$. So $x \in l_D r_D(a^n)$ and hence $l_s r_s(a^n) \subseteq l_D r_D(a^n)$. Take $x = s + t$, where $s \in Da^n$ and $t \in X_{a^n}$. Hence $x = exe = ese\ xete \in eRa^n e + eX_{a^n}e$ and $l_s r_s(a^n) = eDa^n e \oplus eX_{a^n}e = Sa^n \oplus eX_{a^n}$ where eX_{a^n} is aleft ideal of S Therefore, S is right AWNI.■

A ring D is called weak symmetric if abc being nilpotent implies that acb is nilpotent for all $a, b, c \in D$ [12].

Lemma (2.3):[12] If D be a weak symmetric ring, then all nilpotent elements of D are in $J(D)$.

Proposition (2.4): Let D be weak symmetric and right AWNI- ring. Then $J(D) \subseteq Y(D)$.

Proof: Let $0 \neq \alpha \in N(D)$. If $\alpha \notin Y(D)$, then there exists anon zero ideal of D such that $r(\alpha) \cap K = 0$. Hence there exists $b \in K$ such that $\alpha b \neq 0$. Note that $\alpha \in J(D)$(Lem.2.3) and $\alpha b \in J(D)$. Since D

is AWNI-ring, then there exists $0 \neq u \in \alpha bD$ and $n \geq 1$ such that $u^n \neq 0$ and $lr(u^n) = Du^n \oplus X$ for some a left ideal X_{u^n} of D. Write $u^n = \alpha bc$ for some $c \in D$. If $t \in r(\alpha bc)$, then $\alpha bct = 0$ and this implies that $ct \in r(\alpha b) = r(b)$ since $r(\alpha) \cap K = 0$. Hence $(bc)t = b(ct) = 0$ and so $t \in r(bc)$ so $r(bc)= r(\alpha bc)$, Note that $bc \in lr(bc) = lr(\alpha bc) = Du^n \oplus X_{u^n}$. Write $bc = d\alpha bc + x$, where $d \in D$. Hence $x = bc - d\alpha bc$, $x = (1 - d\alpha)bc, x(1 - d\alpha)^{-1} = bc,$ and so $bc \in X_{u^n}$. Since $1 - d\alpha$ is invertible, contradicting with $d\alpha bc \in Du^n - X_{u^n}$. There fore $\alpha \in Y(D)$ and hence $J(D) \subseteq Y(D)$. ■

Lemma (2.5):[2] Let D be a right JCPI. Then $Y(D) \subseteq J(D)$.

From Lem.(2.5) and prop.(2.4), we have the following corollary.

Corollary(2.6): Let D be weak symmetric, right JCPI and right AWNI- ring, then $J(D) = Y(D)$.

Theorem (2.7): If D is a weak symmetric ring whose every simple right D-*module* is AWNI, then D is reduced.

Proof: Assume that $0 \neq a \in D$ with $a^2 = 0$. Thus $r(a) \subseteq L$, where L be amaximal right ideal of D. Since D/L is AWNI, then $l_{D/L}r(a^n) = D/L \oplus X_{a^n}$, $X_{a^n} \leq D/L$. Let $f: aD \longrightarrow D/L$ be defined by $f(ay) = y + L, y \in D$. Note that f is well defined. So $1 + L = f(a) = ca + L, c \in D, x \in X_{a^n}$, by Lem.(2.1) $1 - ca + L = x \in D/L \cap X = 0$, so $1 - ca \in L$, since $ca \in N(D), 1 - ca$ is invertible, which is a contradiction. Therefore, D is a reduced ring. ■

According to [13], a ring D is called GWπ- regular, if for all $a \in J(D)$ there exists $n \geq 1$ such that $a^n \in a^nDa^nD$.

Proposition (2.8): If D is a weak symmetric ring whose every simple right D-*module* is AWNI, then D is GWπ- regular ring.

Proof: Let $a \in N(D)$, then by Lem.(2.3), $a \in J(D)$. We will show that $Da^nD + r(a^n) = D$, $n \geq 1$. If $Da^nD + r(a^n) \neq D$, then there exists amaximal right ideal L of D containing $Da^nD + r(a^n)$. Then by a similar way as in the previous process, there exists $b \in D$ such that $1 - ba^n \in L$. Since $ba^n \in N(D)$, $1 - ba^n$ is invertible, which is a contradiction. There fore $Da^nD + r(a^n) = D$ for any $a \in J(D)$ and implies that D is GWπ- regular ring. ■

From Theorem (2.7) we get corollary

Corollary (2.9): If D is a weak symmetric ring whose every simple right D -*module* is AWNI, then:

1. $N(D) \cap J(D) = 0$
2. If $J(D)$ is nil, then $J(D) = 0$

Proposition (2.10): Let D be a weak symmetric ring, Consequently, the subsequent conditions are similar. D is reduced.

1. D is reduced.
2. D is N-regular.
3. D is a right N-injective.
4. D is a right WN-injective.
5. D is a right AWNI.
6. Every simple right D-*module* is AWNI.

Proof: $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 5 \longrightarrow 6$ is clear and by Theorem (2.7) $6 \longrightarrow 1$ ■

Theorem (2.11): Let D be a weak symmetric and every simple singular right D-*module* is AWNI, then $Y(D) = 0$.

Proof: Suppose that $Y(D) \neq 0$, then $Y(D)$ contains a non zero element z such that $z^2 = 0$. Therefore $r(z) \neq D$. Let L be a maximal right ideal of D containing $r(z)$, then L is an essential right ideal of D which implies that D / L is a right AWNI and $l_{D/L}r(z) = (D/L) \oplus X_z$, $X_z \leq D/L$. Let $f: zD \longrightarrow D/L$ be defined by $f(zy) = y + L$, it is note that f is well defined. Hence, $1 + L = f(z) = cz + L + x, c \in D, x \in X_z, 1 - ca + L = xD/L \cap X = 0$. So $1 - cz \in L$. By Lem.(2.3), $N(D) \subseteq J(D)$, then $cz \in J(D) \subseteq L$. Hence $1 \in L$, contradicting that $L \neq D$.This proves that $Y(D) = 0$. ■

In [13], the trivial extension of any ring D, $D\alpha D = \{(d, c): d, c \in D\}$ with addition defined component vise and multiplication defined by$(d, c)(b, y) = (db, dy + cb)$.

Proposition (2.12): Let D be a ring. If $D\alpha D$ is aright AWNI. Then D is right AGP-injective.

Proof: Let $W = D\alpha D$.For $d \in D$, $(0, d) \in N(W)$. So there exists a left ideal $X_{(0,d)}$ of W and $n \geq 1$ such that $(0, d)^n \neq 0$ and $l_W r_W(0, d)^n = W(0, d)^n \oplus X_{(0,d)^n}$. Since $(0, d)^2 = 0$. It must be that n=1. Hence $l_W r_W(0, d) = W(0, d) \oplus X_{(0,d)}$. By the proof of [14, Prop.(3.1)], if $(b, c) \in l_W r_W(0, d)$ and $(m, n) \in W(0, d)$ then $b = 0, m = 0$. So $X_{(0,d)} = 0\, \alpha X_d$, where $X_d$is a left ideal of D. Again by [14, Prop.(3.1)] we have $l_D r_D(d) = Dd \oplus X_d$. Hence D is right AGP-injective. ■

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